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BIFURCATIONS OF STEADY STATES IN SYSTEMS WITH ROLLING UNDER CONSTANT FORCE PERTURBATIONS†

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Mechanical systems in which the interaction of a rolling elastic body with the supporting plane can be described by well-known axioms [1] are studied. When there are no external forces they belong to dynamical systems with symmetry [2]. A sideways force applied at the centre of mass and the moment of forces about the vertical axis are regarded as symmetry defects. The bifurcation sets of two-, three-, and four-parameter families of steady states are analysed. A procedure for constructing the explicit or parametric forms of the bifurcation surface without constructing the set of steady states is proposed. It is shown that for different values of the control parameters the bifurcation set can have singularities, which can be classified as the cusp, swallow tail, and butterfly catastrophes [3].

1. FORMULATION OF THE PROBLEM

Consider a system consisting of a body with a non-rotating wheel axis and a controlling wheel module whose angle of rotation relative to the body is θ. We will denote by x and y the Cartesian coordinates of the centre of mass C in an inertial frame of reference ϑ being the course angle. We introduce the angular yaw velocity $\omega=\vartheta^{\cdot}$ and the quasivelocities $\upsilon=x^{\cdot}\cos\vartheta+y^{\cdot}\sin\vartheta$, $u=x^{\cdot}\sin\vartheta+y^{\cdot}\cos\vartheta$, which are the longitudinal and transverse projections of the velocity of C. The system is subjected to the sideways reactions Y_i of the supporting plane, which, within the framework of the phenomenological approach [1], are empirical functions of the so-called sideways slipping angle δ_i. In the monotone case the functions $Y_i(\delta_i)$ satisfy the conditions

$$Y_i(0)=0, \quad Y_i(-\delta_i)=-Y_i(\delta_i)$$

$$Y_i'(0)=k_i>0, \quad \lim_{\delta_i\to+\infty} Y_i(\delta_i)/G_i=\varphi_i$$

$$(G_1=mgbl^{-1}, \quad G_2=mgal^{-1}, \quad l=a+b)$$

Let m and I be the mass and central moment of inertia of the system about the vertical axis, a and b, being, respectively, the distances between C and the front and rear wheel axes, $\delta_1=\theta-(u+a\omega)\upsilon^{-1}$, $\delta_2=(-u+b\omega)\upsilon^{-1}$. The equations of the plane-parallel motion of a single-track model with constant velocity υ have the form [4]

$$m(u^{\cdot}+\upsilon\omega)=Y_1\cos\theta+Y_2+Q, \quad I\omega^{\cdot}=aY_1\cos\theta-bY_2+M \tag{1.1}$$

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where Q is the sideways force applied at the centre of mass and M is the moment of forces about the vertical axis.

The singularities of the phase velocity vector field given by (1.1) are solutions of the equations

$$m\upsilon\omega = Y_1(\delta_1)\cos\theta + Y_2(\delta_2) + Q,\ Y_1(\delta_1)a\cos\theta - Y_2(\delta_2)b + M = 0 \tag{1.2}$$

$$(\delta_1 = \delta_1(\omega,\ u,\ \upsilon,\ \theta),\ \delta_2 = \delta_2(\omega,\ u,\ \upsilon)$$

Their number and character depends significantly on υ, θ, Q, and M. For $Q=0$ and $M=0$ a simple bifurcation curve of steady states corresponding to a cusp catastrophe was obtained in the θ, υ-plane [5, 6]. In this case the problem of finding the solution of Eqs (1.2) can, essentially, be reduced [7] to one non-linear equation

$$\upsilon^2(gl)^{-1}(\theta + \delta_2 - \delta_1) = Y(\delta_2 - \delta_1) \tag{1.3}$$

The bifurcation values of θ and υ define the manifold of all tangent lines to the curve $Y = Y(\delta_2 - \delta_1)$, i.e. the dual curve [8]. The cuspidal points of the dual curve correspond to the points of inflection of the original curve.

For $Q=0$ and $M=0$ the system under investigation belongs to the class of dynamical systems with symmetry defined by the conditions

$$x^{\cdot} = f(x, p),\ f(-x, -p) = -f(x, p), x \in R^n, p \in R^m \tag{1.4}$$

The external force fields may cause symmetry defects. In this case

$$x^{\cdot} = f(x, p) + q,\ q \in R_+^n \tag{1.5}$$

2. A METHOD OF CONSTRUCTING THE BIFURCATION SET UNDER CONSTANT FORCE INTERACTIONS

The direct route consists in constructing the set of steady states by a graphical-analytic method [7] or the continuation method with respect to a parameter [4] followed by finding points in these sets where a fusion or birth bifurcation occurs.

Apart from being laborious, the use of this method makes it necessary to know the structure of the bifurcation set, which becomes difficult when the number of control parameters increases. The method proposed below is based on the fact that the bifurcation surface of the steady-state manifold (1.2) is the geometric location of those points in the space control parameters where the "moving straight line" corresponding to a given velocity υ is tangent to the "stationary curve" [7]. Moreover, the values of the bifurcation parameters υ and θ can be determined from the position of the moving line. By traversing the entire stationary curve in this way, we obtain the full bifurcation diagram.

Setting $Y_i^* = Y_i G_i^{-1}$, $Q^* = Q(mg)^{-1}$, and $M^* = Ml(mgab)^{-1}$ and confining ourselves to values of θ such that $\cos\theta \approx 1$, we write (1.2) in the form

$$Y_1^* = \upsilon^2(gl)^{-1}(\theta + \delta_2 - \delta_1) - M^* a l^{-1} - Q^*,\quad Y_1^* = Y_2^* - M^* \tag{2.1}$$

The first equation in (2.1) defines the moving straight line in the plane of $\delta_2 - \delta_1$ and Y_1^*, while the second one defines the stationary curve $Y_1^* = Y_1^*(\delta_2 - \delta_1)$. From the tangency conditions for $Q=0$ we have

$$(Y_1^* - M^* a l^{-1})(\theta + \delta_2 - \delta_1)^{-1} = dY_1^* / d(\delta_2 - \delta_1) \tag{2.2}$$

$$dY_1^* / d(\delta_2 - \delta_1) = \upsilon^2 (gl)^{-1}$$

Hence

$$\theta=(Y_1^*+M^*al^{-1})/(Y_1^*)'-(\delta_2-\delta_1) \tag{2.3}$$

It follows that $\theta=\theta(\delta_2-\delta_1)$. From the second equation in (2.2) we get $\upsilon=\upsilon(\delta_2-\delta_1)$. Therefore (2.2) enables us to find the bifurcation values θ and υ in the form

$$\theta=\theta(\delta_2-\delta_1,\mu),\ \upsilon=\upsilon(\delta_2-\delta_1,\mu),\ M=\mu \tag{2.4}$$

Sometimes (especially for $M\neq 0$) it is useful to take Y_1^* as a parameter. In this case

$$\delta_2-\delta_1=G(Y_1^*) \tag{2.5}$$

should first be found from the second equation in (2.1) [7]. Then $Y_1^*=Y_1^*(\delta_2-\delta_1)$. The function $Y_1^*(\delta_2-\delta_1)$ is the inverse function to $G(Y_1^*)$. The relationships from which to start are $Y_1^*=f_1(\delta_1)$ and $Y_2^*-M^*=f_2(\delta_2)$. Solving them for δ_i, we find $\delta_1=F_1(Y_1^*)$, $\delta_2=F_2(Y_2^*-M^*)$. Therefore $G(Y_1^*)=F_2(Y_1^*)-F_1(Y_1^*)$. It may not always be possible to solve Eq. (2.5) for Y_1^*. In that case, using (2.3) and (2.5) in place of (2.4), from (2.2) we obtain parametric equations of the bifurcation set of the form

$$\theta=(Y_1^*+M^*al^{-1})G'(Y_1^*)-G(Y_1^*)\equiv\theta(Y_1^*) \tag{2.6}$$

$$\upsilon=[gl/G'(Y_1^*)]^{1/2}\equiv\upsilon(Y_1^*),\quad Y_1^*\in(-\varphi_1,\varphi_1)$$

3. REALIZATION OF THE METHOD IN THE SPECIAL CASE OF TWO CONTROL PARAMETERS υ AND θ

Let $Q=0$ and $M=0$. Whether or not it is possible to eliminate the parameter from (2.2) or (2.6) depends not only on the number of control parameters but also on the method of approximating $Y_i(\delta_i)$. In particular, for $Y_i=\beta_i\operatorname{arctg}(\alpha_i\delta_i)$ $(i=1, 2)$, the non-linear function $Y(\delta_2-\delta_1)$ appearing in (1.3) can be obtained analytically. The equation of the stationary curve can be written explicitly as

$$Y=k\operatorname{arctg}[\beta(\delta_2-\delta_1)],\ k=2\varphi\pi^{-1},\ \varphi=\varphi_1=\varphi_2$$

$$\beta=\alpha_1\alpha_2(\alpha_1-\alpha_2)^{-1}$$

Therefore, the equation of the bifurcation curve is

$$\theta=[(1+w^2)\operatorname{arctg}w-w]\beta^{-1} \tag{3.1}$$

$$w=(\upsilon_+^2\upsilon^{-2}-1)^{1/2},\quad \upsilon_+=(k\beta gl)^{1/2}$$

Here υ_+ is the critical velocity of rectilinear motion.

For $w<1$ we find from (3.1) that

$$\theta=2(3\beta)^{-1}w^3+O(w^5),\quad \theta\approx 2(3\beta)^{-1}(\upsilon_+^2\upsilon^{-2}-1)^{3/2} \tag{3.2}$$

It follows that the curve (3.1) has a cusp point of the first kind for $\theta=0$ and $\upsilon=\upsilon_+$. In a small neighbourhood of the tip it can be approximated by a semicubical parabola. The curve (3.1) is represented by a dashed curve in Fig. 1 for $m-2717$ kg, $l=4061$ kg m^2, and $a=1.14$ m, $b=1.6$ m, $k_1=1177200$ N/rad, $k_2=103500$ N/rad, $\varphi=0.8$, $\upsilon=8$ m/s. One of the properties of (1.4) is reflected in the symmetry of the curve about the straight line $\theta=0$.

The cuspidal point $(0, \upsilon_+)$ of the bifurcation set can be a safe or dangerous [9] boundary of the stability domain in the space of parameters (the matrix of the corresponding system of linearized equations of perturbed motion can have one zero eigenvalue). This property may be connected [4] with a real fusion or birth bifurcation from the origin of the frame of reference in the phase space of a pair of singular points:

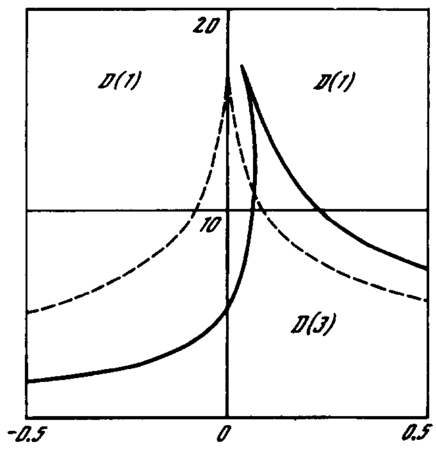

Fig. 1.

the safe boundary corresponds to a birth bifurcation, and the dangerous one to a fusion bifurcation. Therefore, according to the geometrical interpretation of the governing equation (1.3), a condition for the boundary to be safe or, respectively, dangerous, is that the graph of $Y(\delta_2-\delta_1)$ should be convex or concave in the neighbourhood of zero.

It is possible for the type of convexity to be altered as the parameters of the system are changed with the points of inflection arriving at the origin (a higher-order inflection is then obtained at zero) and the corresponding point $(0,\ \upsilon_+)$ of the bifurcation set develops a singularity of higher rank.

Let us analyse this possibility for the approximation

$$Y_i(\delta_i)=k_i\delta_i(1+\kappa_i^2\varphi_i^{-2}\delta_i^2)^{-1/2}, \quad \kappa_i=k_iG_i^{-1}$$

Since in this case it is impossible to represent $Y(\delta_2-\delta_1)$ analytically, we shall analyse the convexity of the inverse function $G(Y)=\delta_2-\delta_1$. We find that

$$G(Y)=(\kappa_2^{-1}-\kappa_1^{-1})Y+\tfrac{1}{2}(\kappa_2^{-1}\varphi_2^{-2}-\kappa_1^{-1}\varphi_1^{-2})Y^3+\tfrac{3}{8}(\kappa_2^{-1}\varphi_2^{-4}-\kappa_1^{-1}\varphi_1^{-4})Y^5+\ldots \tag{3.3}$$

Here $\kappa_1>\kappa_2$: there is a critical velocity υ_+ with $\upsilon_+^2=\kappa_1\kappa_2gl(\kappa_1-\kappa_2)^{-1}$. If the graph of $G(Y)$ lies above the tangent line at zero, the curve $Y=Y(\delta_2-\delta_1)$ is concave. If it lies below the tangent line, the curve is convex. It follows from (3.3) that the boundary is dangerous for $\varphi_1^2>\kappa_2\kappa_1^{-1}\varphi_2^2$. The condition $\varphi_1^2\ll\kappa_2\kappa_1^{-1}\varphi_2^2$ ensures that the boundary is safe (for $\varphi_1^2=\kappa_2\kappa_1^{-1}\varphi_2^2$ the coefficient in the third term in (3.3) is negative). Since

$$G(Y)=\kappa_2^{-1}Y(1-\varphi_2^{-2}Y^2)^{-1/2}-\kappa_1^{-1}Y(1-\varphi_1^2Y^2)^{-1/2}$$

$$G''(Y)=0\Rightarrow Y_0^2=(\kappa_1^{2/5}\varphi_1^{4/5}-\kappa_2^{2/5}\varphi_2^{4/5})(\kappa_1^{2/5}\varphi_1^{-6/5}-\kappa_2^{2/5}\varphi_2^{-6/5})^{-1}$$

it is confirmed that for $\varphi_2\kappa_2^{1/2}\kappa_1^{-1/2}<\varphi_1<\varphi_2$ two points of inflection exist in the vicinity of the origin ($Y_0^2>0$). For $\varphi_1^2=\kappa_2\kappa_1^{-1}\varphi_2^2$ there is a higher-order inflection at the origin. It follows that a "butterfly" singularity is realized at the corresponding point of the bifurcation set, turning the cusp into a double cusp. This process is illustrated in Fig. 2, in which $\varphi_2=0.8$. The presence of three cuspidal points in the bifurcation set is due to the fact that to each point of inflection of the stationary curve there corresponds a cuspidal point on the bifurcation curve.

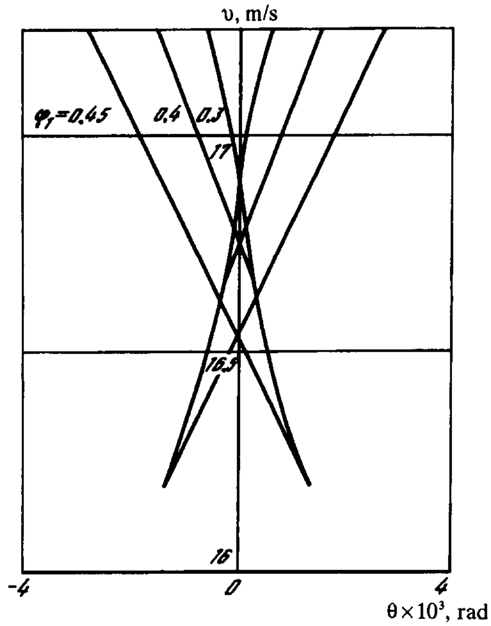

Fig. 2.

4. THE EFFECT OF THE SIDEWAYS FORCE ($M=0$)

It follows from (2.1) that for $Q>0$ the moving straight line can be dropped on to Q^*. This is equivalent to turning the front wheels by an angle $\theta_* = -Q^* g/\upsilon^{-2}$. The maximum value of the sideways force for which a stable steady state exists is

$$Q^*_{\max} = |k\operatorname{arctg}(\upsilon_+^2\upsilon^{-2}-1)^{1/2} - \beta\upsilon^2(gl)^{-1}(\upsilon_+^2\upsilon^{-2}-1)^{1/2}|$$

The bifurcation set can be described by

$$\theta = Q^* gl\upsilon^{-2} + \beta^{-1}[(1+w^2)\operatorname{arctg} w - w] \tag{4.1}$$

which can be obtained from (3.1) by replacing θ by $\theta - Q^* g/\upsilon^{-2}$. In Fig. 1 we show the sections of the surface (4.1) by the planes $Q^*=0$ (the dashed line) and $Q^*=0.5$ (the solid line). Their configuration indicates that the sideways force leads to substantial quantitative changes without altering the qualitative character of the bifurcation set. In particular, $\upsilon_+=16.9$ m/s for $Q^*=0$ while $Q^*=0.5$ m/s for $\upsilon_+=5.1$. By turning the front wheels through the appropriate angle in the appropriate direction, one can eliminate the effect of the sideways force and obtain the desired type of motion.

In the domain $D(3)$ there is one stable steady state and two unstable ones. In $D(1)$ there is one unstable state. The spike corresponds to a three-fold singular point in the phase plane, indicating a cusp catastrophe on the steady-state manifold.

The positive parameters Q and M break symmetry. In their presence the system belongs to the class (1.5) and the bifurcation set is asymmetric about the axis $\theta=0$.

5. THE EFFECT OF THE MOMENT OF FORCES ($Q=0$)

In this case only the parametric representation of the bifurcation surface is possible, in general. In Fig. 3 we present the cross-sections of the surface (2.4) by the planes $M^*=0.12$ (the solid line), $M^*=0.24$ (the

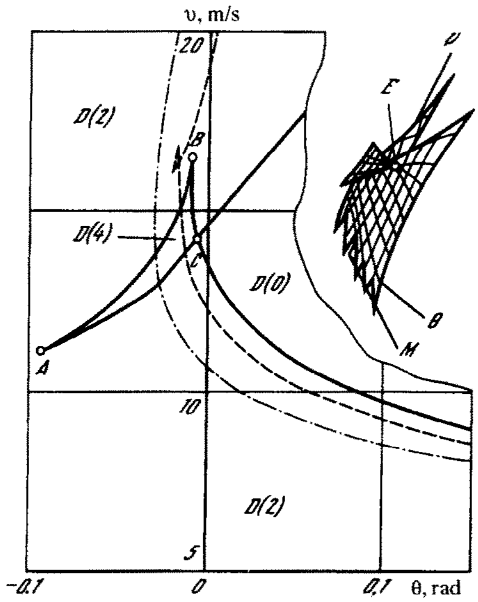

Fig. 3.

dashed line), and $M^* = 0.4$ (the dash–dot line) for the above-mentioned numerical values. The bifurcation diagram indicates that when M varies a swallow tail catastrophe is obtained in the three-dimensional space of control parameters θ, υ and M. We denote the domains in which there are k singular points by $D(k)$ where $k = 0, 2, 4$. On crossing the boundary of the bifurcation set the number of points decreases or increases by two. The points A, B and C correspond to the tips of the cusp and are triple singular points. The domain between the edges corresponds to four singular points. For a certain value of M (in Fig. 3 it lies in the range $0.24 < M^* < 0.4$) a four-fold singular point is obtained, corresponding to a swallow tail singularity. The general form of the bifurcation surface is shown on the right of Fig. 3. The point E gives rise to a four-fold singular point in phase space. The presence of this point indicates a swallow tail catastrophe.

6. CONCLUSION

Within the framework of the phenomenological approach [1], for systems with rolling, with the classical monotone dependence $Y_i(\delta_i)$ and $M = 0$, $Q = 0$ the governing curve $Y = Y(\delta_2 - \delta_1)$ has a simple point of inflection at zero and the bifurcation set has a cuspidal point at $(0, \upsilon_+)$ (the singularity A_3 is a "fold" [8]). In the neighbourhood of the cusp point the bifurcation set is defined by (3.2). Aperiodic loss of steady-state stability occurs at the points $(\theta(\upsilon), \upsilon)$, corresponding to circular trajectories in the support plane. The critical velocity for circular regimes decreases as a semicubic function of θ.

The presence of the moment of forces $M < mgabl^{-1}(\varphi_1 + \varphi_2)$ gives rise to one more cuspidal point in the bifurcation set, which merges with the first one at $(\theta_*, \upsilon_*, M_*)$ (the singularity A_4 is a "swallow tail" [8]). At the corresponding point the function $Y(\delta_2 - \delta_1, M) \equiv Y_1^*(\delta_2 - \delta_1)$ has a higher-order inflection and the equation $\upsilon^2(gl)^{-1}(\theta + \delta_2 - \delta_1) - M^*al^{-1} = Y(\delta_2 - \delta_1, M)$ has a four-fold root. The bifurcation values of the parameters form a surface dividing the space into three domains with different numbers of steady states 4, 2, and 0.

Hence, only cuspidal stable singularities can be found in the mechanical systems under consideration: a cusp in the family of steady states with two parameters (υ, θ), a swallow tail in the case of three-parameters (υ, θ, M) and a butterfly in the case of four parameters $(\upsilon, \theta, \varphi_1, \varphi_2)$.

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